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Eigenfrequencies and modified eigenmodes of a rectangular plate with free edges☆

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ABSTRACT

A solution of the problem of the oscillations of a rectangular plate with free edges, which enables exact values of the eigenfrequencies and approximate eigenmodes to be obtained, is given. The use of the results obtained to design plates on an elastic foundation is proposed. The calculation is carried out by the Ritz method in which the eigenmodes of the plate obtained are taken as the coordinate functions, which considerably speeds up the process of obtaining a solution compared with existing approaches.

Using the linear technical theory of the bending of plates [1] we consider the flexural natural oscillations of a rectangular Plate ($-a \le x \le a$, $-b \le y \le b$) with free edges having a cylindrical stiffness D and Poisson's ratio of the material ν_p. The equation of flexural oscillations has the form

$$(\Delta^2 - \lambda)W = 0, \quad \lambda = m\omega^2/D \tag{1}$$

The static boundary conditions are[1,2]

$$\begin{aligned} x = \pm a: \frac{\partial^2 W}{\partial x^2} + \nu_p \frac{\partial^2 W}{\partial y^2} = \frac{\partial^3 W}{\partial x^3} + (2-\nu_p)\frac{\partial^3 W}{\partial x \partial y^2} = 0 \\ y = \pm b: \frac{\partial^2 W}{\partial y^2} + \nu_p \frac{\partial^2 W}{\partial x^2} = \frac{\partial^3 W}{\partial y^3} + (2-\nu_p)\frac{\partial^3 W}{\partial x^2 \partial y} = 0 \end{aligned} \tag{2}$$

where $W(x,y)$ are the deflections of the plate, m is the distributed mass and ω is the eigenfreqency.

Without loss of generality, we will consider oscillations that are symmetrical about the x and y axes. We will change to the dimensionless variables $\bar{x} = x/a$, $\bar{y} = y/b$. Henceforth, omitting the bar over x and y, we will represent the deflections of the plate in the form of the sum of two particular solutions[3] of the equation $\Delta\Delta W - \lambda W = 0$:

$$W(x,y) = C_1 \cos\alpha x \cos\beta y + C_2 \operatorname{ch}\alpha x \operatorname{ch}\beta y \tag{3}$$

Substituting expression (3) into boundary conditions (2) and expanding the determinant, we obtain the following transcendental equations for determining α and β, well-known from the theory of beam functions[4]

$$\chi^+(\alpha) = 0, \quad \chi^+(\beta) = 0; \quad \chi^\pm(\gamma) = \operatorname{th}\gamma \pm \operatorname{tg}\gamma, \quad \gamma = \alpha, \beta \tag{4}$$

Substituting the solutions of Eqs (4) into the equation of oscillations (1) we can write the following expression for the eigenfrequencies, symmetrical about the x and y axes, in the form well-known from the reference books[1,5]

$$\omega_{ik} = \left(\frac{\alpha_i^2}{a^2} + \frac{\beta_k^2}{b^2}\right)\sqrt{\frac{D}{m}} \tag{5}$$

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Table 1

Characteristics of the oscillations	Eigenfunctions	Transcendental equations and roots
Symmetrical about the *x* and *y* axes	$\cos\,\alpha_i x\cos\beta_k y-\frac{\sin\alpha_i\sin\beta_k}{\mathrm{sh}\,\alpha_i\mathrm{sh}\beta_k}\mathrm{ch}\,\alpha_i x\mathrm{ch}\,\beta_k y$	$x^+=(\alpha_i)=0$ $x^+=(\beta_k)=0$ $\alpha_1=0=\beta_1$ $\alpha_2=2.3650=\beta_2$ $\alpha_3=5.4978=\beta_3$ . . .
Symmetricals about the *y* axis	$\cos\,\alpha_i x\sin\beta_k y-\frac{\sin\alpha_i\cos\beta_k}{\mathrm{sh}\,\alpha_i\mathrm{ch}\beta_k}\mathrm{ch}\,\alpha_i x\mathrm{sh}\,\beta_k y$	$x^+(\alpha_i)=0$ $x^-(\beta_k)=0$ $\beta_1=0$ $\beta_2=3.9267$ $\beta_3=7.0686$ . . .
Symmetricals about the *x* axis	$\sin\,\alpha_i x\cos\beta_k y-\frac{\cos\alpha_i\sin\beta_k}{\mathrm{ch}\,\alpha_i\mathrm{sh}\beta_k}\mathrm{sh}\,\alpha_i x\mathrm{ch}\,\beta_k y$	$x^-(\alpha_i)=0$ $x^+(\beta_k)=0$
Antisymmetricals about the *x* and *y* axes	$\sin\,\alpha_i x\sin\beta_k y-\frac{\cos\alpha_i\cos\beta_k}{\mathrm{ch}\,\alpha_i\mathrm{ch}\,\beta_k}\mathrm{sh}\,\alpha_i x\mathrm{sh}\,\beta_k y$	$x^-(\alpha_i)=0$ $x^-(\beta_k)=0$

To determine the eigenmodes we will use the condition for the torques at the corners of a rectangular plate with free edges to be equal to zero[6]

$$x=\pm 1,\quad y=\pm 1:\frac{\partial^2 W}{\partial x\partial y}=0 \tag{6}$$

From condition (3) this leads to the expression

$$\frac{C_2}{C_1}=-\frac{\sin\alpha\sin\beta}{\mathrm{sh}\alpha\mathrm{sh}\beta}$$

which enables us to represent the *ik*-mode, symmetrical about the *x* and *y* coordinate axes, in the form

$$W_{ik}(x,y)=C_1\left(\cos\alpha_i x\cos\beta_k y-\frac{\sin\alpha_i\sin\beta_k}{\mathrm{sh}\alpha_i\mathrm{sh}\beta_k}\mathrm{ch}\alpha_i x\mathrm{ch}\beta_k y\right) \tag{7}$$

The eigenfrequencies and eigenmodes, symmetrical and antisymmetrical about one and two coordinate axes, can be obtained similarly.

In Table 1 we present expressions for the eigenmodes and the forms of the transcendental equations for determining the eigenfrequencies using formula (5). In Fig. 1 we show three eigenmodes of flexural oscillations of a rectangular plate for $a=1$ m and $b=2$ m. It should be borne in mind that the first eigemodes of a rectangular plate with free edges correspond to the values $\alpha=\beta=0$ and to displacements of the plate as a rigid body. In Table 2 we compare the values obtained for the dimensionless eigenfrequencies of a square plate $\bar{\omega}_k=\omega_k\sqrt{ma^4/D}(k=2,3,4)$ with Ritz's results[7] and the Kantorovich–Krylov results.[8]

Note the following.

1°. The eigenfrequencies and eigenmodes obtained are independent of Poisson's ratio of the plate material.

2°. A breakdown in the orthogonality properties of the eigenmodes obtained is observed, and the value of the error decreases as the frequency corresponding to this mode increases. Thus, for a square plate, the integral

$$\int_{-1}^{1}\int_{-1}^{-1}W_{ii}(x,y)W_{kk}(x,y)dxdy$$

representing the orthogonality property, is equal to zero when $i=1$ and $k=2, 3, 4$, is equal to 0.017812 when $i=2$, $k=3$, is equal to 0.0044 when $i=2$ and $k=4$, and is equal to 0.00821 when $i=3$ and $k=4$.

3°. For a rectangular plate with free edges, the eigenfrequencies, defined by (5), are obviously obtained exactly. However, it is not possible to determine the eigenmode exactly in this case using the technical theory of the bending of plates, since it is necessary to satisfy exactly three boundary conditions on the free edge of the plate,[9] which, in the approximate theory of the bending of plates are replaced by two conditions, that combine the action of the transverse force and the toque due to the reduced transverse force. These forces are integrally in equilibrium and, according to the Saint-Venant principle, they correspond to forces which decrease rapidly from the plate edge, but in this case the property of orthogonality of the modes obtained breaks down.

Table 2

Frequencies	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$
Characteristic of the oscillation	Symmetrical about the *x* and *y* axes	Symmetrical about the *y* axis and Antisymmetrical about the *x* axis	Antisymmetrical about the *x* and *y* axes
According to Ritz[7]	14.10	20.56	23.91
Kantorovich–Krylov[8]	12.43	–	–
Results of calculations	11.19	21.91	30.83

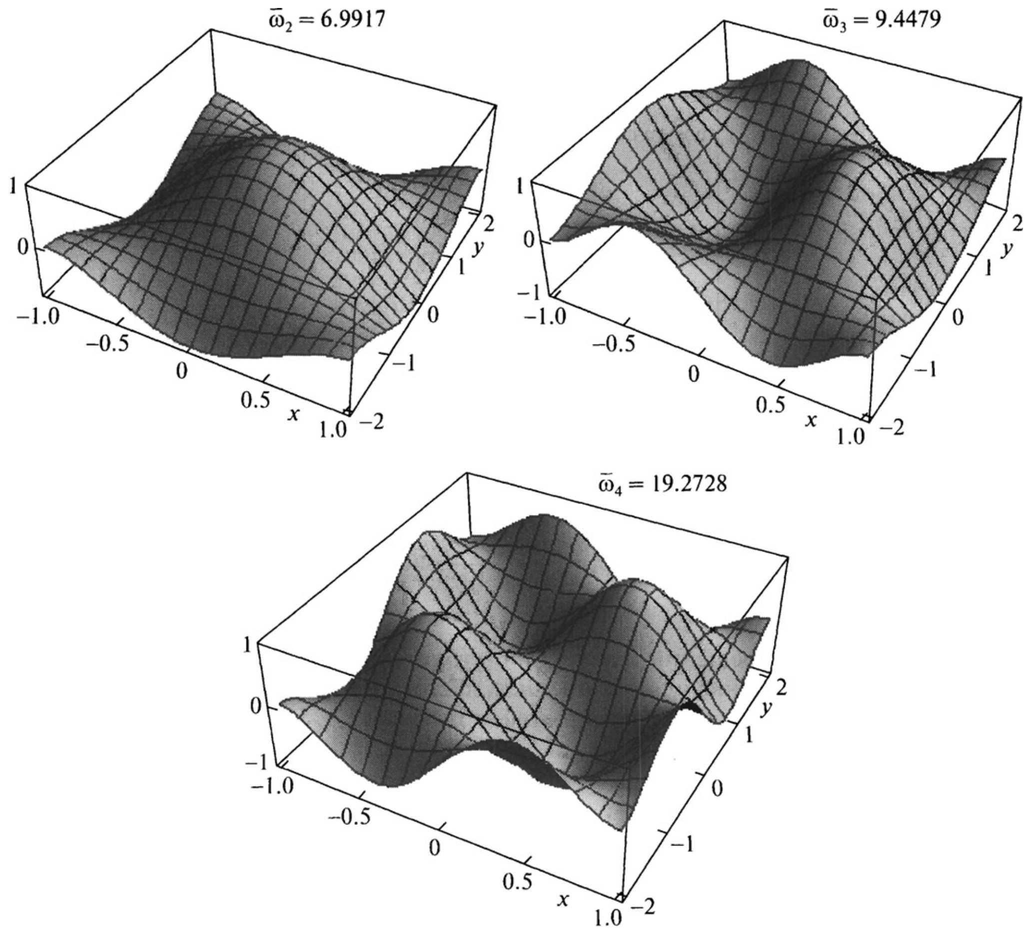

Fig. 1.

We will consider the use of the results obtained for a static calculation of a rectangular plate on a Winkler foundation acted upon by a centrally applied concentrated force. The equilibrium equation of the plate has the form[10]

$$(\Delta^2 + k/D)W = q(x,y)/D \tag{8}$$

where q(x,y) is the vertical load on the plate and k is the stiffness coefficient of the Winkler foundation.

We will specify the equation of the plate deflection in the form (7)

$$W(x,y) = C_{11} + \sum_{i=2}^{\infty}\sum_{k=2}^{\infty} C_{ik}W_{ik}(x,y) \tag{9}$$

Substituting expression (9) into Eq. (8) and differentiating, we obtain the equality

$$\frac{ka^4}{D}C_{11} + \sum_{i=2}^{\infty}\sum_{k=2}^{\infty}\left[\left(\alpha_i^2 + \frac{a^2}{b^2}\beta_k^2\right)^2 + \frac{ka^4}{D}\right]C_{ik}W_{ik}(x,y) = q(x,y)\frac{a^4}{D} \tag{10}$$

We multiply both sides of this equation by $\frac{T_{2m}(x)T_{2n}(y)}{\sqrt{1-x^2}\sqrt{1-y^2}}$, where $T_{2m}(z)$ is a Chebyshev polynomial of the first kind,[11] and we integrate over x from −1 to +1 and over y from −1 to +1. In this case the concentrated force is distributed over the area of an infinitesimal rectangle at the origin of coordinates. We obtain a system of linear algebraic equations in the unknown C_{ik}. It can be shown,[8] that the system obtained is regular and the set of free terms has an upper limit, and hence it can be solved by the truncation method.

For example, we will consider a reinforced concrete rectangular plate of width a = 1 m, length b = 2 m and thickness 0.4 m on a Winkler foundation with a stiffness coefficient $k = 2 \times 10^7$ N/m^3. We obtain $ka^4/D = 0.1215$ and

$$\bar{C}_{11} = 73.9961,\quad \bar{C}_{22} = -10.9069,\quad \bar{C}_{23} = -6.3887,\quad \bar{C}_{32} = -1.1053,$$
$$\bar{C}_{33} = -1.4041\ldots\quad \left(\bar{C}_{ik} = \left(\frac{Pa^3}{\pi^2 bD}\right)^{-1} C_{ik}\right)$$

In Fig. 2 we show the deflection surface of this rectangular plate on a Winkler foundation.

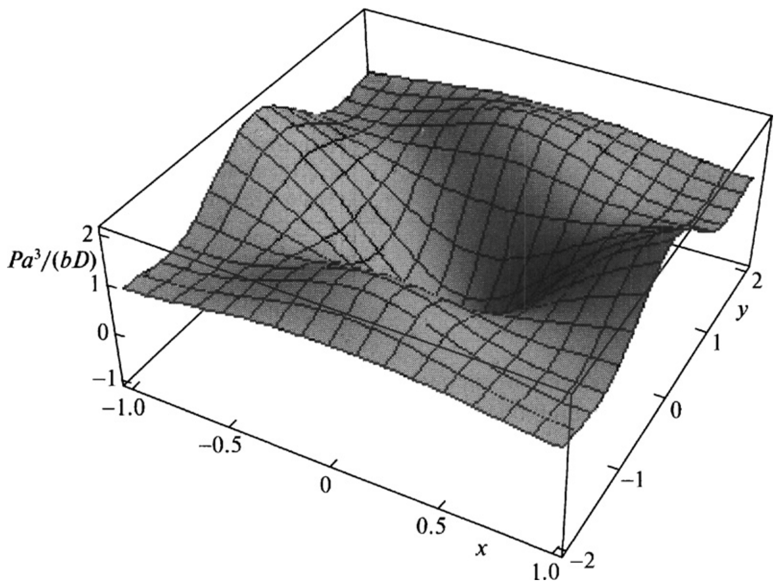

Fig. 2.

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